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Hydrodynamics of giant planet formation

I. Overviewing the κ-mechanism

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ABSTRACT

Context. To investigate the physical nature of the 'nucleated instability' of proto giant planets, the stability of layers in static, radiative gas spheres is analysed on the basis of Baker's standard one-zone model.
Aims. It is shown that stability depends only upon the equations of state, the opacities and the local thermodynamic state in the layer. Stability and instability can therefore be expressed in the form of stability equations of state which are universal for a given composition.
Methods. The stability equations of state are calculated for solar composition and are displayed in the domain $-14 \leq \lg \rho/[\mathrm{g\,cm^{-3}}] \leq 0$, $8.8 \leq \lg e/[\mathrm{erg\,g^{-1}}] \leq 17.7$. These displays may be used to determine the one-zone stability of layers in stellar or planetary structure models by directly reading off the value of the stability equations for the thermodynamic state of these layers, specified by state quantities as density ρ, temperature T or specific internal energy e. Regions of instability in the (ρ, e)-plane are described and related to the underlying microphysical processes.
Results. Vibrational instability is found to be a common phenomenon at temperatures lower than the second He ionisation zone. The κ-mechanism is widespread under 'cool' conditions.

Key words. giant planet formation – κ-mechanism – stability of gas spheres

1. Introduction

In the *nucleated instability* (also called core instability) hypothesis of giant planet formation, a critical mass for static core envelope protoplanets has been found. Mizuno (1980) determined the critical mass of the core to be about $12\,M_\oplus$ ($M_\oplus = 5.975 \times 10^{27}$ g is the Earth mass), which is independent of the outer boundary conditions and therefore independent of the location in the solar nebula. This critical value for the core mass corresponds closely to the cores of today's giant planets.

Although no hydrodynamical study has been available many workers conjectured that a collapse or rapid contraction will ensue after accumulating the critical mass. The main motivation for this article is to investigate the stability of the static envelope at the critical mass. With this aim the local, linear stability of static radiative gas spheres is investigated on the basis of Baker's (1966) standard one-zone model.

Phenomena similar to the ones described above for giant planet formation have been found in hydrodynamical models concerning star formation where protostellar cores explode (Tscharnuter 1987, Balluch 1988), whereas earlier studies found quasi-steady collapse flows. The similarities in the (micro)physics, i.e., constitutive relations of protostellar cores and protogiant planets serve as a further motivation for this study.

⋆ Just to show the usage of the elements in the author field
⋆⋆ The university of heaven temporarily does not accept e-mails

2. Baker's standard one-zone model

In this section the one-zone model of Baker (1966), originally used to study the Cepheïd pulsation mechanism, will be briefly reviewed. The resulting stability criteria will be rewritten in terms of local state variables, local timescales and constitutive relations.

Baker (1966) investigates the stability of thin layers in self-gravitating, spherical gas clouds with the following properties:

- hydrostatic equilibrium,
- thermal equilibrium,
- energy transport by grey radiation diffusion.

For the one-zone-model Baker obtains necessary conditions for dynamical, secular and vibrational (or pulsational) stability (Eqs. (34a, b, c) in Baker 1966). Using Baker's notation:

M_r mass internal to the radius r
m mass of the zone
r_0 unperturbed zone radius
ρ_0 unperturbed density in the zone
T_0 unperturbed temperature in the zone
L_{r0} unperturbed luminosity
E_{th} thermal energy of the zone

and with the definitions of the *local cooling time* (see Fig. 1)

$$\tau_{\mathrm{co}} = \frac{E_{\mathrm{th}}}{L_{r0}}, \tag{1}$$

Fig. 1. Adiabatic exponent Γ_1. Γ_1 is plotted as a function of lg internal energy [erg g^{-1}] and lg density [g cm^{-3}].

and the *local free-fall time*

$$\tau_{\rm ff} = \sqrt{\frac{3\pi}{32G}\frac{4\pi r_0^3}{3M_{\rm r}}}, \tag{2}$$

Baker's K and σ_0 have the following form:

$$\sigma_0 = \frac{\pi}{\sqrt{8}}\frac{1}{\tau_{\rm ff}} \tag{3}$$

$$K = \frac{\sqrt{32}}{\pi}\frac{1}{\delta}\frac{\tau_{\rm ff}}{\tau_{\rm co}}; \tag{4}$$

where $E_{\rm th} \approx m(P_0/\rho_0)$ has been used and

$$\begin{array}{l}\delta = -\left(\frac{\partial\ln\rho}{\partial\ln T}\right)_P \\ e = mc^2\end{array} \tag{5}$$

is a thermodynamical quantity which is of order 1 and equal to 1 for nonreacting mixtures of classical perfect gases. The physical meaning of σ_0 and K is clearly visible in the equations above. σ_0 represents a frequency of the order one per free-fall time. K is proportional to the ratio of the free-fall time and the cooling time. Substituting into Baker's criteria, using thermodynamic identities and definitions of thermodynamic quantities,

$$\Gamma_1 = \left(\frac{\partial\ln P}{\partial\ln\rho}\right)_S, \ \chi_\rho = \left(\frac{\partial\ln P}{\partial\ln\rho}\right)_T, \ \kappa_P = \left(\frac{\partial\ln\kappa}{\partial\ln P}\right)_T$$

$$\nabla_{\rm ad} = \left(\frac{\partial\ln T}{\partial\ln P}\right)_S, \ \chi_T = \left(\frac{\partial\ln P}{\partial\ln T}\right)_\rho, \ \kappa_T = \left(\frac{\partial\ln\kappa}{\partial\ln T}\right)_T$$

one obtains, after some pages of algebra, the conditions for *stability* given below:

$$\frac{\pi^2}{8}\frac{1}{\tau_{\rm ff}^2}(3\Gamma_1 - 4) > 0 \tag{6}$$

$$\frac{\pi^2}{\tau_{\rm co}\tau_{\rm ff}^2}\Gamma_1\nabla_{\rm ad}\left[\frac{1-3/4\chi_\rho}{\chi_T}(\kappa_T - 4) + \kappa_P + 1\right] > 0 \tag{7}$$

$$\frac{\pi^2}{4}\frac{3}{\tau_{\rm co}\tau_{\rm ff}^2}\Gamma_1^2\,\nabla_{\rm ad}\left[4\nabla_{\rm ad} - (\nabla_{\rm ad}\kappa_T + \kappa_P) - \frac{4}{3\Gamma_1}\right] > 0 \tag{8}$$

For a physical discussion of the stability criteria see Baker (1966) or Cox (1980).

We observe that these criteria for dynamical, secular and vibrational stability, respectively, can be factorized into

1. a factor containing local timescales only,
2. a factor containing only constitutive relations and their derivatives.

The first factors, depending on only timescales, are positive by definition. The signs of the left hand sides of the inequalities (6), (7) and (8) therefore depend exclusively on the second factors containing the constitutive relations. Since they depend only on state variables, the stability criteria themselves are *functions of the thermodynamic state in the local zone*. The one-zone stability can therefore be determined from a simple equation of state, given for example, as a function of density and temperature. Once the microphysics, i.e. the thermodynamics and opacities (see Table 1), are specified (in practice by specifying a chemical composition) the one-zone stability can be inferred if the thermodynamic state is specified. The zone – or in other words the layer – will be stable or unstable in whatever object it is imbedded as long as it satisfies the one-zone-model assumptions. Only the specific growth rates (depending upon the time scales) will be different for layers in different objects.

Table 1. Opacity sources.

Source	T/[K]
Yorke 1979, Yorke 1980a	≤ 1700[a]
Krügel 1971	$1700 \leq T \leq 5000$
Cox & Stewart 1969	$5000 \leq$

[a] This is footnote a

We will now write down the sign (and therefore stability) determining parts of the left-hand sides of the inequalities (6), (7) and (8) and thereby obtain *stability equations of state*.

The sign determining part of inequality (6) is $3\Gamma_1 - 4$ and it reduces to the criterion for dynamical stability

$$\Gamma_1 > \frac{4}{3}\cdot \tag{9}$$

Stability of the thermodynamical equilibrium demands

$$\chi_\rho > 0, \quad c_v > 0, \tag{10}$$

and

$$\chi_T > 0 \tag{11}$$

holds for a wide range of physical situations. With

$$\Gamma_3 - 1 = \frac{P}{\rho T}\frac{\chi_T}{c_v} > 0 \tag{12}$$

$$\Gamma_1 = \chi_\rho + \chi_T(\Gamma_3 - 1) > 0 \tag{13}$$

$$\nabla_{\rm ad} = \frac{\Gamma_3 - 1}{\Gamma_1} > 0 \tag{14}$$

we find the sign determining terms in inequalities (7) and (8) respectively and obtain the following form of the criteria for dynamical, secular and vibrational *stability*, respectively:

$$3\Gamma_1 - 4 =: S_{\rm dyn} > 0 \tag{15}$$

$$\frac{1-3/4\chi_\rho}{\chi_T}(\kappa_T - 4) + \kappa_P + 1 =: S_{\rm sec} > 0 \tag{16}$$

$$4\nabla_{\rm ad} - (\nabla_{\rm ad}\kappa_T + \kappa_P) - \frac{4}{3\Gamma_1} =: S_{\rm vib} > 0. \tag{17}$$

The constitutive relations are to be evaluated for the unperturbed thermodynamic state (say (ρ_0, T_0)) of the zone. We see that the one-zone stability of the layer depends only on the constitutive relations Γ_1, $\nabla_{\rm ad}$, χ_T, χ_ρ, κ_P, κ_T. These depend only on the unperturbed thermodynamical state of the layer. Therefore the above relations define the one-zone-stability equations of state $S_{\rm dyn}$, $S_{\rm sec}$ and $S_{\rm vib}$. See Fig. 2 for a picture of $S_{\rm vib}$. Regions of secular instability are listed in Table 1.

Fig. 2. Vibrational stability equation of state $S_{\mathrm{vib}}(\lg e, \lg \rho)$. > 0 means vibrational stability.

3. Conclusions

1. The conditions for the stability of static, radiative layers in gas spheres, as described by Baker's (1966) standard one-zone model, can be expressed as stability equations of state. These stability equations of state depend only on the local thermodynamic state of the layer.
2. If the constitutive relations – equations of state and Rosseland mean opacities – are specified, the stability equations of state can be evaluated without specifying properties of the layer.
3. For solar composition gas the κ-mechanism is working in the regions of the ice and dust features in the opacities, the H_2 dissociation and the combined H, first He ionization zone, as indicated by vibrational instability. These regions of instability are much larger in extent and degree of instability than the second He ionization zone that drives the Cepheïd pulsations.

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References

Baker, N. 1966, in Stellar Evolution, ed. R. F. Stein,& A. G. W. Cameron (Plenum, New York) 333
Balluch, M. 1988, A&A, 200, 58
Cox, J. P. 1980, Theory of Stellar Pulsation (Princeton University Press, Princeton) 165
Cox, A. N.,& Stewart, J. N. 1969, Academia Nauk, Scientific Information 15, 1
Mizuno H. 1980, Prog. Theor. Phys., 64, 544
Tscharnuter W. M. 1987, A&A, 188, 55
Terlevich, R. 1992, in ASP Conf. Ser. 31, Relationships between Active Galactic Nuclei and Starburst Galaxies, ed. A. V. Filippenko, 13
Yorke, H. W. 1980a, A&A, 86, 286
Zheng, W., Davidsen, A. F., Tytler, D. & Kriss, G. A. 1997, preprint